

Tiling by lattices for locally compact abelian groups

Davide Barbieri, Eugenio Hernández, Azita Mayeli

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Abstract

For a locally compact abelian group G a simple proof is given for the known fact that a bounded domain Ω tiles G with translations by a lattice Λ if and only if the set of characters of G indexed by the dual lattice of Λ is an orthogonal basis of $L^2(\Omega)$. The proof uses simple techniques from Harmonic Analysis.

1 Introduction

Let G denote a locally compact and second countable abelian group. A closed subgroup Λ of G is called a *lattice* if it is discrete and co-compact, i.e, the quotient group G/Λ is compact. Since G is second countable, then any discrete subgroup of G is also countable [21, Section 12, Example 17]. We denote the dual group by \widehat{G} . The dual lattice of Λ is defined as follows:

$$\Lambda^\perp = \{\xi \in \widehat{G} : \langle \xi, \lambda \rangle = 1 \ \forall \lambda \in \Lambda\}, \quad (1)$$

where $\langle \xi, \lambda \rangle$ indicates the action of character ξ on the group element λ .

The dual lattice Λ^\perp is a subgroup of \widehat{G} and by the duality theorem between subgroups and quotient groups [23, Lemma 2.1.2], it is topologically isomorphic to the dual group of G/Λ , i.e., $\Lambda^\perp \cong \widehat{G/\Lambda}$. Since G/Λ is compact, then by the isomorphic relation, the dual lattice Λ^\perp is discrete. Notice that also by [23, Lemma 2.1.2], $\widehat{G/\Lambda^\perp} \cong \widehat{\Lambda}$. This implies that Λ^\perp is co-compact, hence it is a lattice.

Let dg denote a Haar measure on G . For a function f in $L^1(G)$ the Fourier transform of f is defined by

$$\mathcal{F}_G(f)(\chi) = \int_G f(g) \overline{\langle \chi, g \rangle} dg, \quad \chi \in \widehat{G},$$

where $\langle \chi, g \rangle$ denotes the action of the character χ on g . By the inversion theorem [23, Section 1.5.1], a Haar measure $d\chi$ can be chosen on \widehat{G} so that the Fourier transform \mathcal{F}_G is an isometry from $L^2(G)$ onto $L^2(\widehat{G})$.

For any $\chi \in \widehat{G}$, we define the *exponential function* e_χ by

$$e_\chi : G \rightarrow \mathbb{C}, \quad e_\chi(g) := \langle \chi, g \rangle.$$

For any subset Ω of G , we let $|\Omega|$ denote the Haar measure of Ω . Throughout this paper, we let $\mathbf{1}_\Omega$ denote the characteristic function of set Ω . We shall also use addition ‘+’ for the group action since G is abelian.

Definition 1.1 (Tiling). *We say a subset $\Omega \subset G$ with positive and non-zero Haar measure, tiles G with the translations by a lattice Λ of G if for almost every $g \in G$ there is only one $\lambda \in \Lambda$ and only one $x \in \Omega$ such that g has the representation $g = x + \lambda$. Notice that the uniqueness of the representations implies that for any two lattice points $\lambda_1 \neq \lambda_2$, the sets $\Omega + \lambda_1$ and $\Omega + \lambda_2$ do not intersect up to a measure zero set. In this case we say (Ω, Λ) is a tiling pair. Equivalently, (Ω, Λ) is a tiling pair if*

$$\sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(g - \lambda) = 1 \quad \text{a.e. } g \in G. \quad (2)$$

Definition 1.2 (Spectrum). *Let $\tilde{\Lambda}$ be a countable subset of \widehat{G} . We say $\tilde{\Lambda}$ is a spectrum for Ω , if the exponentials $\{e_{\tilde{\lambda}} : \tilde{\lambda} \in \tilde{\Lambda}\}$ form an orthogonal basis for $L^2(\Omega)$.*

In the sequel, we will assume that Ω has positive measure and Λ is a lattice in G . Therefore, the annihilator Λ^\perp is also a lattice in \widehat{G} . Our main result is the following.

Theorem 1.3 (Main Result). *Let Λ and Ω be as above. Then the following are equivalent:*

- (1) *The set Ω tiles G with translations by the lattice Λ .*
- (2) *$|\Omega| = |Q_\Lambda|$, and the system of translations $\{\sqrt{|\Omega|}^{-1} \mathbf{1}_\Omega(\cdot - \lambda) : \lambda \in \Lambda\}$ is an orthonormal system in $L^2(G)$.*
- (3) *$|\Omega| = |Q_\Lambda|$, and*

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\mathbf{1}_\Omega)(\chi + \tilde{\lambda})|^2 = |\Omega|^2 \quad \text{a.e. } \chi \in \widehat{G}.$$

- (4) *For all $f \in L^2(G)$,*

$$\|f \mathbf{1}_\Omega\|_{L^2(G)}^2 = |\Omega|^{-1} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(f \mathbf{1}_\Omega)(\tilde{\lambda})|^2.$$

- (5) *The exponential set $\{e_{\tilde{\lambda}} : \tilde{\lambda} \in \Lambda^\perp\}$ is an orthogonal basis for $L^2(\Omega)$.*

The study of the relationship between spectrum sets and tiling pairs has its roots in a 1974 paper of B. Fuglede ([6]) where he proved that a set $E \subset \mathbb{R}^d$, $d \geq 1$, of positive Lebesgue measure, tiles \mathbb{R}^d by translations with a lattice Λ if and only if $L^2(E)$ has an orthogonal basis of exponentials indexed by the annihilator of Λ .

The general statement in \mathbb{R}^d , which says that if $E \subset \mathbb{R}^d$, $d \geq 1$, has positive Lebesgue measure, then $L^2(E)$ has an orthogonal basis of exponentials (not necessary indexed by a lattice) if and only if E tiles \mathbb{R}^d by translations, has been known as the Fuglede Conjecture. A variety of results were proved establishing connections between tiling and orthogonal exponential bases. See, for example, [19], [13], [18], [14] and [15]. In 2001, I. Laba proved that the Fuglede conjecture is true for the union of two intervals in the plane ([17]). In 2003, A. Iosevich, N. Katz and T. Tao ([11]) proved that the Fuglede conjecture holds for convex planar domains. The conjecture was also proved for the unit cube of \mathbb{R}^d in [13] and [19].

In 2004 Terry Tao ([24]) disproved the Fuglede Conjecture in dimension $d = 5$ and larger, by exhibiting a spectral set in \mathbb{R}^5 which does not tile the space by translations. In [16], M. Kolountzakis and M. Matolcsi also disproved the reverse implication of the Fuglede Conjecture for dimensions $d = 4$ and larger. In [3] and [2], the dimension of counter-examples was further reduced. In fact, B. Farkas, M. Matolcsi and P. Mora show in [2] that the Fuglede conjecture is false in \mathbb{R}^3 . The general feeling in the field is that sooner or later the counter-examples of both implications will cover all dimensions. However, in [12] the authors showed that the Fuglede Conjecture holds in two-dimensional vector spaces over prime fields.

The extension of B. Fuglede result for lattices in \mathbb{R}^d , $d \geq 1$, to second countable LCA groups has been proved by S. Pedersen in [20], where mainly topological arguments are used to prove that (1) and (5) of Theorem 1.3 are equivalent. In the current paper, analytical methods are used instead to prove this result, while other equivalent conditions are given. We emphasize that the equivalence of (2) and (3) in Theorem 1.3 comes from the theory of dual integrable representations developed in [8] (see Section 1.1 for details). The authors strongly believe that the techniques used in this paper are suggestive and can lead to more discoveries on the hidden relations between translations and exponential bases in general.

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1.1 Notations and Preliminaries

Let Λ be a lattice in a second countable LCA group G . Denote by $Q_\Lambda \subset G$ a measurable cross section of G/Λ . By definition, a cross section is a set of representatives of all cosets in G/Λ such that the intersection of Q_Λ with any coset $g + \Lambda$ has only one element. Its existence is guaranteed by [4, Theorem 1]. A cross section is called *fundamental domain* if it is relatively compact. In our situation, every cross section is a fundamental domain since G/Λ is compact. By the definition, it is evident that (Q_Λ, Λ) is a tiling pair for G . For a lattice Λ , the *lattice size* is defined as the Haar measure of a fundamental domain Q_Λ , i.e., $|Q_\Lambda|$, therefore the measure of any fundamental domain.

Let $d\dot{g}$ be a normalized Haar measure for G/Λ . Then the relation between Haar measure on G and Haar measure for G/Λ is given by the following *Weil's formula*: for any function $f \in L^1(G)$, the periodization map $\Phi(\dot{g}) = \sum_{\lambda \in \Lambda} f(g + \lambda)$, $\dot{g} \in G/\Lambda$ is well defined almost everywhere in G/Λ , belongs to $L^1(G/\Lambda)$, and

$$\int_G f(g) dg = |Q_\Lambda| \int_{G/\Lambda} \sum_{\lambda \in \Lambda} f(g + \lambda) d\dot{g}. \quad (3)$$

This formula is a especial case of [22, Theorem 3.4.6]. The constant $|Q_\Lambda|$ appears in 3 because G/Λ is equipped with the normalized Haar measure.

Here, we shall recall *Poisson summation formula* (see [7, Lemma 6.2.2] or [5, Theorem 4.42]).

Theorem 1.4. *Given $f \in L^1(G)$ and a lattice Λ in G , if $\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(f)(\tilde{\lambda})|^2 < \infty$, then the periodization map Φ , given before (3), belongs to $L^2(G/\Lambda)$ and*

$$\sum_{\lambda \in \Lambda} f(g + \lambda) = |Q_\Lambda|^{-1} \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(f)(\tilde{\lambda}) e_{\tilde{\lambda}}(g) \quad a.e. \ g \in G. \quad (4)$$

Here, both series converge in $L^2(G/\Lambda)$. Moreover, if $f \in C_c(G)$, i.e, continuous and compactly supported, then Poisson Summation formula (4) holds pointwise ([5]).

For any function $f : G \rightarrow \mathbb{C}$, we define $\tilde{f}(g) := \overline{f(-g)}$. And for any given two functions f and h defined on G , the convolution $f * h$ is given by

$$f * h(g) = \int_G f(g_1) h(g - g_1) dg_1,$$

provided that the integral exists.

Definition 1.5 (Dual integrable representations ([8])). Let G be an LCA group, and let π be a unitary representation of G on a Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle$. We say π is dual integrable if there exists a Haar measure $d\chi$ on \widehat{G} and a function $[\cdot, \cdot]_\pi : \mathcal{H} \times \mathcal{H} \rightarrow L^1(\widehat{G})$, called bracket map for π , such that for all $\phi, \psi \in \mathcal{H}$

$$\langle \phi, \pi(g)\psi \rangle = \int_{\widehat{G}} [\phi, \psi]_\pi(\chi) e_{-g}(\chi) d\chi \quad \forall g \in G.$$

Example 1.6. Let Λ be a lattice in a second countable LCA group G . Let $\phi \in L^2(G)$. For any $\lambda \in \Lambda$, define $T_\lambda(\phi)(g) = \phi(g - \lambda)$. We show that T is a dual integrable representation. Let $M_\lambda h(\chi) := e_\lambda(\chi)h(\chi)$, and let Q_{Λ^\perp} be a fundamental domain for the annihilator lattice Λ^\perp . Thus, by an application of Parseval identity and Weil's formula (3) we have

$$\begin{aligned} \langle \phi, T_\gamma \psi \rangle &= \langle \mathcal{F}_G(\phi), M_\gamma \mathcal{F}_G(\psi) \rangle \\ &= \int_{\widehat{G}} \mathcal{F}_G(\phi)(\chi) \overline{\mathcal{F}_G(\psi)(\chi)} e_{-\gamma}(\chi) d\chi \\ &= |Q_{\Lambda^\perp}| \int_{\widehat{G}/\Lambda^\perp} \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(\phi)(\chi + \tilde{\lambda}) \overline{\mathcal{F}_G(\psi)(\chi + \tilde{\lambda})} e_{-\lambda}(\chi + \tilde{\lambda}) d\dot{\chi} \\ &= |Q_{\Lambda^\perp}| \int_{\widehat{G}/\Lambda^\perp} \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(\phi)(\chi + \tilde{\lambda}) \overline{\mathcal{F}_G(\psi)(\chi + \tilde{\lambda})} e_{-\lambda}(\chi) d\dot{\chi}. \end{aligned} \tag{5}$$

Now take

$$[\phi, \psi]_T(\dot{\chi}) := |Q_{\Lambda^\perp}| \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(\phi)(\chi + \tilde{\lambda}) \overline{\mathcal{F}_G(\psi)(\chi + \tilde{\lambda})} \quad \text{a.e. } \dot{\chi} \in \widehat{G}/\Lambda^\perp.$$

Then, T is a dual integrable representation with $\mathcal{H} = L^2(G)$ and the bracket function $[\cdot, \cdot]_T$, which belongs to $L^1(\widehat{G}/\Lambda^\perp)$ because $\mathcal{F}_G(\phi) \overline{\mathcal{F}_G(\psi)} \in L^1(\widehat{G})$.

One of practical application of dual integrable representations is that one can characterize bases in terms of their associated bracket functions. For example, the following result has been given in [8, Proposition 5.1].

Theorem 1.7. Let G be a countable abelian group and π be a dual integrable unitary representation of G on a Hilbert space \mathcal{H} . Let $\phi \in \mathcal{H}$. Then the system $\{\pi(g)\phi : g \in G\}$ is an orthonormal basis for its spanned vector space if and only if $[\phi, \phi]_\pi(\chi) = 1$ for almost every $\chi \in \widehat{G}$.

As a byproduct of the preceding theorem we have the following result.

Corollary 1.8. Let T and Λ be as in Example 1.6. Let $\phi \in L^2(G)$. Then the translations system $\{T_\lambda \phi : \lambda \in \Lambda\}$ is an orthonormal system in $L^2(G)$ if and only if

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\phi)(\chi + \tilde{\lambda})|^2 = |Q_{\Lambda^\perp}|^{-1} \quad \text{a.e. } \chi \in \widehat{G}.$$

We need a couple of Lemmata that will also be used in the proof of Theorem 1.3.

Lemma 1.9. *Let Ω tile G by lattice Λ . Let Q_Λ denote a fundamental domain for Λ . Then $|\Omega| = |Q_\Lambda|$.*

Proof. Let $f = \mathbf{1}_\Omega$ be the characteristic function of Ω . By Weil's formula (3) we have:

$$|\Omega| = \int_G \mathbf{1}_\Omega(g) dg = |Q_\Lambda| \int_{G/\Lambda} \sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(g + \gamma) d\dot{g} \quad (6)$$

By the assumption, Ω tiles G by Λ -translations. Therefore,

$$\sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(g + \gamma) = 1, \quad a.e. g \in G.$$

Thus

$$|\Omega| = |Q_\Lambda| \int_{G/\Lambda} 1 d\dot{g} = |Q_\Lambda|, \quad (7)$$

since $d\dot{g}$ is a normalized Haar measure for G/Λ . This proves our lemma. \square

We shall recall the following lemma whose proof has been shown in [7, Lemma 6.2.3].

Lemma 1.10. *Let Λ be a lattice in G . Then the annihilator Λ^\perp is a lattice in \widehat{G} and*

$$|Q_\Lambda| |Q_{\Lambda^\perp}| = 1.$$

2 Proof of main Theorem - Theorem 1.3

In the rest, we shall prove Theorem 1.3.

Proof. “(1) \Leftrightarrow (2)”. The implication “(1) \Rightarrow (2)” is trivial since Ω tiles G by Λ -translations. We now prove “(2) \Rightarrow (1)”. By the orthogonality of translations, for any distinct λ_1 and $\lambda_2 \in \Lambda$,

$$0 = \langle \mathbf{1}_\Omega(\cdot - \lambda_1), \mathbf{1}_\Omega(\cdot - \lambda_2) \rangle_{L^2(G)} = |\Omega + \lambda_1 \cap \Omega + \lambda_2|.$$

This indicates that the translations of Ω by $\lambda_1 \neq \lambda_2$ have an intersection whose measure is zero. It remains to show that the all λ -translations of Ω , $\lambda \in \Lambda$, cover the whole group G almost everywhere. Let $F := \dot{\cup}_{\lambda \in \Lambda} \Omega + \lambda$, where the $\dot{\cup}$ is used to denote almost disjoint union. Then

$$\sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(x - \lambda) = \mathbf{1}_F(x) \quad a.e. x \in F.$$

Let $E = G \setminus F$. We shall prove that $|E| = 0$. Let Q_Λ be any fundamental set for Λ . We prove that $|F \cap Q_\Lambda| = |Q_\Lambda|$:

$$\begin{aligned}
|F \cap Q_\Lambda| &= \int_G \mathbf{1}_{F \cap Q_\Lambda}(x) dx = \int_G \mathbf{1}_{Q_\Lambda}(x) \mathbf{1}_F(x) dx \\
&= \int_G \mathbf{1}_{Q_\Lambda}(x) \left(\sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(x - \lambda) \right) dx \\
&= \sum_{\lambda \in \Lambda} \int_{Q_\Lambda} \mathbf{1}_\Omega(x - \lambda) dx \\
&= \sum_{\lambda \in \Lambda} \int_{\lambda + Q_\Lambda} \mathbf{1}_\Omega(y) dy \\
&= \int_G \mathbf{1}_\Omega(y) dy = |\Omega| = |Q_\Lambda|.
\end{aligned} \tag{8}$$

From the above calculations, we conclude that $F \cap Q_\Lambda = Q_\Lambda$ up to a set of measure zero, since Q_Λ has finite measure. Similarly, one can show that for all $\lambda \in \Lambda$, $F \cap (Q_\Lambda + \lambda) = Q_\Lambda + \lambda$ up to a measure zero set. Since G is a countable union of the pairwise disjoint sets $Q_\Lambda + \lambda, \lambda \in \Lambda$, the set $E = G \setminus F$ is a disjoint countable union of sets of measure zero. Hence, E is a set of measure zero. This completes the proof.

“(2) \Leftrightarrow (3)”. Follows from Corollary 1.8 and Lemmata 1.9 and 1.10.

“(4) \Leftrightarrow (5)”. The implication “(5) \Rightarrow (4)” is Parseval identity for the exponential basis $\{e_{\tilde{\lambda}} \mathbf{1}_\Omega : \tilde{\lambda} \in \Lambda^\perp\}$. Let us prove now “(4) \Rightarrow (5)”. Let $h \in L^2(\Omega)$ be such that $\langle h, e_{\tilde{\lambda}} \mathbf{1}_\Omega \rangle = 0$ for all $\tilde{\lambda} \in \Lambda^\perp$. Then

$$\mathcal{F}_G(h \mathbf{1}_\Omega)(\tilde{\lambda}) = \int_\Omega h(g) \overline{e_{\tilde{\lambda}}(g)} dg = \langle h, e_{\tilde{\lambda}} \mathbf{1}_\Omega \rangle = 0.$$

Therefore, by the equation in condition (4), $h = 0$ a.e.. This proves the completeness of the exponentials $\{e_{\tilde{\lambda}}\}_{\Lambda^\perp}$ in $L^2(\Omega)$. To show that the exponentials are orthogonal, fix $\tilde{\lambda}_0 \in \Lambda^\perp$ and put $f(g) := e_{\tilde{\lambda}_0}(g) \mathbf{1}_\Omega(g)$. Once again, by condition (4) we have

$$\begin{aligned}
|\Omega| &= \|f\|_{L^2(G)}^2 = |\Omega|^{-1} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(e_{\tilde{\lambda}_0} \mathbf{1}_\Omega)(\tilde{\lambda})|^2 \\
&= |\Omega|^{-1} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle e_{\tilde{\lambda}_0}, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)}|^2 \\
&= |\Omega| + |\Omega|^{-1} \sum_{\tilde{\lambda} \neq \tilde{\lambda}_0} |\langle e_{\tilde{\lambda}_0}, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)}|^2.
\end{aligned}$$

This implies that

$$\langle e_{\tilde{\lambda}_0}, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)} = 0 \quad \forall \tilde{\lambda} \neq \tilde{\lambda}_0,$$

and this takes care of the orthogonality of the exponentials.

“(4) \Rightarrow (3)”. Fix $\chi \in \widehat{G}$ and define $f(g) = \overline{\chi(g)}\mathbf{1}_\Omega(g)$, $g \in G$. Then we have

$$\mathcal{F}_G(f)(\tilde{\lambda}) = \mathcal{F}_G(\mathbf{1}_\Omega)(\tilde{\lambda} + \chi), \quad \forall \tilde{\lambda} \in \Lambda^\perp.$$

Thus for f , by the condition (4), we obtain

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\mathbf{1}_\Omega)(\tilde{\lambda} + \chi)|^2 = |\Omega| \|\overline{\chi(g)}\mathbf{1}_\Omega\|^2 = |\Omega|^2.$$

“(1) \Rightarrow (4)”. Let $f \in C_c(G)$, be continuous and compactly supported. We can write the following:

$$\begin{aligned} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(f\mathbf{1}_\Omega)(\tilde{\lambda})|^2 &= \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(f\mathbf{1}_\Omega)(\tilde{\lambda}) \overline{\mathcal{F}_G(f\mathbf{1}_\Omega)(\tilde{\lambda})} \\ &= \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(f\mathbf{1}_\Omega)(\tilde{\lambda}) \mathcal{F}_G(\widetilde{f\mathbf{1}_\Omega})(\tilde{\lambda}) \\ &= \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(f\mathbf{1}_\Omega * \widetilde{f\mathbf{1}_\Omega})(\tilde{\lambda}). \end{aligned}$$

Since $f \in C_c(G)$, then $f\mathbf{1}_\Omega * \widetilde{f\mathbf{1}_\Omega} \in C_c(G)$. Then Poisson summation formula (see Theorem 1.4) for $f\mathbf{1}_\Omega * \widetilde{f\mathbf{1}_\Omega}$ at $g = e$ implies that

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(f\mathbf{1}_\Omega * \widetilde{f\mathbf{1}_\Omega})(\tilde{\lambda}) = |\Omega| \sum_{\lambda \in \Lambda} (f\mathbf{1}_\Omega * \widetilde{f\mathbf{1}_\Omega})(\lambda).$$

Therefore,

$$\begin{aligned} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(f\mathbf{1}_\Omega)(\tilde{\lambda})|^2 &= |\Omega| \sum_{\lambda \in \Lambda} (f\mathbf{1}_\Omega * \widetilde{f\mathbf{1}_\Omega})(\lambda) \\ &= |\Omega| \sum_{\lambda \in \Lambda} \int_G (f\mathbf{1}_\Omega)(g) (\widetilde{f\mathbf{1}_\Omega})(\lambda - g) dg \\ &= |\Omega| \sum_{\lambda \in \Lambda} \int_G (f\mathbf{1}_\Omega)(g) \overline{(f\mathbf{1}_\Omega)(g - \lambda)} dg \\ &= |\Omega| \sum_{\lambda \in \Lambda} \int_{\Omega \cap \Omega + \lambda} f(g) \overline{f(g - \lambda)} dg. \end{aligned}$$

By the assumption (1) of the theorem, $|\Omega \cap \Omega + \lambda| = 0$ for $\lambda \neq e$. Therefore,

$$|\Omega|^{-1} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(f\mathbf{1}_\Omega)(\tilde{\lambda})|^2 = \int_\Omega f(g) \overline{f}(g) dg = \int_G |f\mathbf{1}_\Omega(g)|^2 dg = \|f\mathbf{1}_\Omega\|_{L^2(G)}^2.$$

This shows (4) for $f \in C_c(G)$. Use a density argument to prove the result for general $f \in L^2(G)$. □

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D. Barbieri, Universidad Autónoma de Madrid, 28049 Madrid, Spain.
dave.barbieri@uam.es

E. Hernández, Universidad Autónoma de Madrid, 28049 Madrid, Spain.
eugenio.hernandez@uam.es

A. Mayeli, City University of New York, Queensborough.
amayeli@gc.cuny.edu